Journal of the Technical University of Gabrovo

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DESIGN OF SYNCHRONIZATION SCHEMES WITH IDENTICAL, ANTI- AND HYBRID SYNCHRONIZATION FOR A FIFTH-ORDER CHAOTIC MODEL

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ARTICLE INFO

Article history:
Received 21 October 2021
Accepted 9 December 2021

Keywords:
chaotic systems; chaotic synchronization; Lyapunov stability theory

ABSTRACT

The subject of this paper is the design of chaotic synchronization schemes with different types of synchronization for a fifth-order chaotic model. Chaotic systems are a specific class on nonlinear systems with complex dynamics, which are extremely sensitive to the initial conditions and possess a strange attractor in the state space, consisting of infinite number of unstable periodic orbits. Two or more such systems can synchronize their chaotic behavior, if a suitable control function is designed and applied to them. These tasks are very popular research field in the last two decades, because synchronized chaotic systems can be used for data protection in secure communication systems or for file encryption. In this paper, chaotic synchronization schemes with identical synchronization, with anti-synchronization and with hybrid synchronization are designed on the basis of a fifth-order chaotic model. The control functions are calculated using the Lyapunov's second stability method.

1. INTRODUCTION

The phenomenon called chaotic synchronization was first discovered by Pecora and Carroll [1-3], who proved that two or more chaotic systems can synchronize their dynamics by means of a coupling signal between them. Chaotic systems are nonlinear systems that are usually described by simple differential equations. Their characteristic feature is their strong sensitivity to their initial conditions and to changes in some of their parameters, as a result of which their dynamics can change dramatically and from an initially steady state they can obtain a complex attractor in the state space in which there is a random factor. Such an attractor is called chaotic attractor.

The synchronization of such systems has great potential and some real applications in various technical and other scientific fields. This is especially true for the fields of secure communications, data encryption, system identification etc. [4-10].

There are different types of chaotic synchronization, the most common of which is the identical chaotic synchronization [11-12], in which two or more chaotic systems are coupled in such a way as to perform identical chaotic movements. Other types of synchronization are less studied, such as the anti-synchronization of chaotic systems [13-14], in which the dynamics of the variables of one chaotic system is the same by module, but with the opposite sign, compared to the dynamics of the variables of the other system. By the so called hybrid synchronization [15-16], some of the respective variable pairs of the two chaotic systems to be synchronized are in identical synchronization mode, and the other variable pairs are in anti-synchronization mode. In principle, the use of chaotic synchronization systems with more complex types of synchronization than the identical one gives greater opportunities for information protection in communication systems with chaotic data protection.

Usually, a chaotic model, which has been previously studied and its characteristic features are well known, is chosen as the basis of a given synchronization system. The predominant part of the known continuous chaotic models are of the third order [17-18]. Continuous nonlinear systems of lower than third order cannot generate chaos. A much smaller number of known chaotic models are of the fourth order [19-20]. Chaotic models of higher than fourth order [21-22] are an insignificant percentage of the total number of known chaotic models. At the same time, such high-order models have more complex dynamics, which is also of interest in view of the potential applications of chaotic synchronization schemes based on high-order chaotic models in the field of secure communications.

This paper deals with a little-known model of a fifth-order chaotic system described by Navier-Stokes equations. Based on this model, chaotic synchronization schemes with control functions, calculated using the second Lyapunov stability method, are realized, in which identical synchronization, anti-synchronization and hybrid synchronization are successively obtained.

EXPOSITION

When dealing with a specific hydrodynamic system presented in [23], an interesting mode of work has been

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found in which the system has a strange attractor and, accordingly, chaotic behavior. A similar system is presented in [24]. For the purposes of the present work, it is not the physical nature of the system that is of interest, but the equations of its model itself, which will be considered as an abstract chaotic fifth-order generator. The equations of the model belong to the class of systems described by Navier-Stokes equations, and in dimensionless form they are:

$$\begin{aligned}
\dot{x}_1 &= a_1 x_1 + a_2 x_2 x_3 + a_3 x_4 x_5,\\
\dot{x}_2 &= a_4 x_2 + a_5 x_1 x_3,\\
\dot{x}_3 &= a_6 x_3 + a_7 x_1 x_2 + r,\\
\dot{x}_4 &= a_8 x_4 - x_1 x_5,\\
\dot{x}_5 &= -x_5 + a_9 x_1 x_4,
\end{aligned} \tag{1}$$

where the nominal values of the system parameters are: $a_1 = -2$, $a_2 = 4$, $a_3 = 4$, $a_4 = -9$, $a_5 = 3$, $a_6 = -5$, $a_7 = -7$, $a_8 = -5$, $a_9 = -3$. The parameter r is a bifurcation parameter, i.e. when it changes, the dynamics of the system change qualitatively. In [23] the equilibrium states of the nonlinear model (1) are studied, where the following has been found:

1. For $0 \le r \le R_1$, where $R_1 = 5\sqrt{3/2}$, the system (1) has only one equilibrium point, which is stable and in fact is a global attractor of the system for values of r in this range. The coordinates of this point are:

$$\bar{x}_1 = \bar{x}_2 = \bar{x}_4 = \bar{x}_5 = 0,\ \bar{x}_3 = r/5 .$$

2. For $R_1 \le r \le R_2$, where $R_2 = (80/9)\sqrt{3/2}$, the system (1) has three equilibrium points, the first of which is the same as in the previous case 1., but is already unstable, and the other two points are stable.

3. For $r > R_2$, the system (1) has seven equilibrium points, the first three are those of the previous case 2, but are already unstable, and the other four equilibrium points are stable, but only for $r < R_3 = 22.853$. When $r = R_3$ these points become unstable too and four periodic orbits are formed around them. As the value of the parameter r increases further, the number of orbits progressively doubles by a cascade of bifurcations until a transition to a chaotic state is reached. This is achieved for $r \ge 33$.

From the point of view of the problem of chaotic synchronization, the dynamics of the system for $r \ge 33$ is of interest. The system (1) is simulated in the Simulink simulation environment of the Matlab software product for $r = 33$ and the nominal values of the other parameters, given above. The initial conditions are $\mathbf{x}(t_0) = \mathbf{x}(0) = \begin{bmatrix} 5 & 4 & 1 & 1 & 1 \end{bmatrix}^T$ and are chosen randomly. The simulation shows type of dynamics which is typical for chaotic systems. Fig. 1 shows four three-dimensional projections of the five-dimensional chaotic attractor in the subspaces of the state space, respectively (x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_3, x_4) and (x_2, x_3, x_5). The selected projections give an idea of the structure of the attractor of the system - a large number of closely spaced periodic orbits, which is the main feature of chaotic systems. The transition from one orbit to another is random. In Fig. 2 two of the many two-dimensional projections of the attractor are shown, respectively in the planes (x_1, x_3) and (x_1, x_4). Two-dimensional projections can give a more detailed view of the location of the individual orbits of the attractor. Fig. 3 shows the time series of some of the state variables - $x_1(t)$ and $x_3(t)$. The dynamics of the other variables of the system is similar and in general it resembles the development over time of a random process. Therefore, only the time diagrams of the variables of a system cannot prove whether it is chaotic or stochastic. The selected simulation parameters are: simulation time 40 s, integration method ode4 (Runge-Kutta) and a fixed step with a size of 0.01 s.

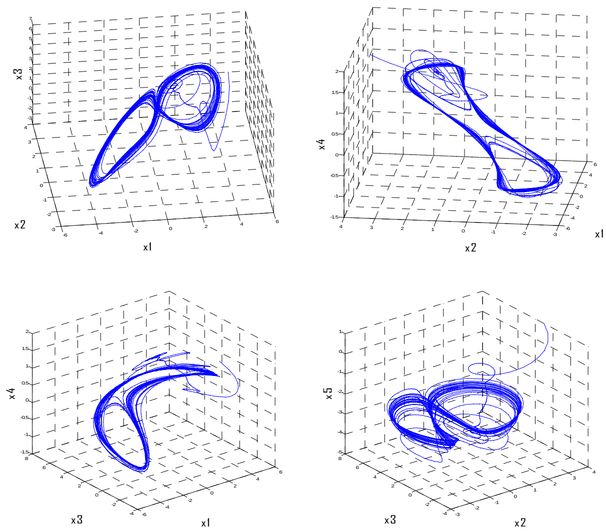

Fig. 1. Projections of the five-dimensional attractor in different three-dimensional subspaces of the state space

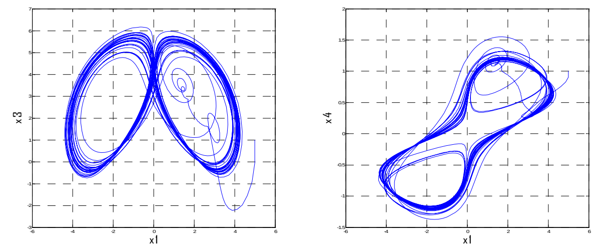

Fig. 2. Selected two-dimensional projections of the chaotic attractor

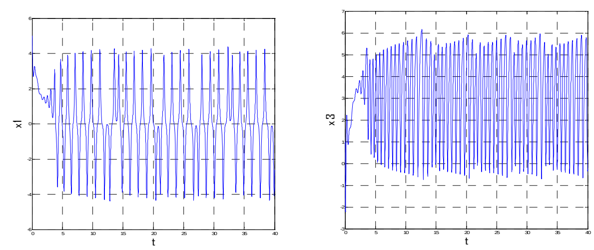

Fig. 3. Time series of some of the state variables

Let the system (1) be represented in the generalized form:

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t)\right), \tag{2}$$

where $\mathbf{x} \in \Re^5$ is the state vector of the system, i.e. $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T$, and the function $\mathbf{f}$ contains the right parts of (1).

The synthesis of a chaotic synchronization scheme requires a second instance of the same chaotic system (1), which can be written in the form:

$$\begin{aligned}
\dot{\tilde{x}}_1 &= a_1\tilde{x}_1 + a_2\tilde{x}_2\tilde{x}_3 + a_3\tilde{x}_4\tilde{x}_5 + g_1(\mathbf{x},\tilde{\mathbf{x}}),\\
\dot{\tilde{x}}_2 &= a_4\tilde{x}_2 + a_5\tilde{x}_1\tilde{x}_3 + g_2(\mathbf{x},\tilde{\mathbf{x}}),\\
\dot{\tilde{x}}_3 &= a_6\tilde{x}_3 + a_7\tilde{x}_1\tilde{x}_2 + r + g_3(\mathbf{x},\tilde{\mathbf{x}}),\\
\dot{\tilde{x}}_4 &= a_8\tilde{x}_4 - \tilde{x}_1\tilde{x}_5 + g_4(\mathbf{x},\tilde{\mathbf{x}}),\\
\dot{\tilde{x}}_5 &= -\tilde{x}_5 + a_9\tilde{x}_1\tilde{x}_4 + g_5(\mathbf{x},\tilde{\mathbf{x}}),
\end{aligned} \tag{3}$$

where the state variables of the second system are denoted with $\tilde{x}_i, i = 1 \div 5$, and $g_i(\mathbf{x},\tilde{\mathbf{x}})$ are control functions, which are subject to determination depending on the set goals.

The pair of systems (1) - (3) will represent a chaotic synchronization scheme/system, if such control functions $g_i(\mathbf{x},\tilde{\mathbf{x}})$ are found that the systems (1) and (3) to be in the mode of identical, anti-, hybrid or other type of chaotic synchronization. A general rule for this type of tasks is that the two chaotic systems are identical ones, with the same set of parameters, but with different initial conditions. The nature of the coupling between the two chaotic systems is such that variables from the system (1) participate in the control functions to the system (3), but the variables from the system (3) do not participate in the equations of (1), i.e. the coupling is an one-way one. Then the system (1) is called a control or a master system, and the system (3) - a controlled or a slave system. Similar to (2), the equations of the slave system can be written in compact form:

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{f}(\tilde{\mathbf{x}}(t)) + \mathbf{g}(\mathbf{x}(t),\tilde{\mathbf{x}}(t)), \tag{4}$$

where $\tilde{\mathbf{x}} \in \Re^5$, т.е.: $\tilde{\mathbf{x}} = [\tilde{x}_1 \quad \tilde{x}_2 \quad \tilde{x}_3 \quad \tilde{x}_4 \quad \tilde{x}_5]^T$, and the vector $\mathbf{g}$ contains the control functions to the slave system $g_i(\mathbf{x},\tilde{\mathbf{x}})$.

The goal of chaotic synchronization tasks is to synthesize the functions $g_i(\mathbf{x},\tilde{\mathbf{x}})$ in such way that the following condition is fulfilled:

$$\lim_{t\to\infty} \mathbf{e}(t) = 0, \tag{5}$$

where $\mathbf{e} \in \Re^n$ is the vector with the error functions between the different pairs of variables of the master and the slave systems, which in the case of identical synchronization between these systems has the form:

$$\mathbf{e}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t), \tag{6}$$

and in the case of anti-synchronization is:

$$\mathbf{e}(t) = \mathbf{x}(t) + \tilde{\mathbf{x}}(t), \tag{7}$$

i.e. in the case of **identical synchronization**, when all functions $e_i = x_i - \tilde{x}_i, i = 1 \div 5$ settle to zero, the systems, for example (1) and (3), will perform identical chaotic movements, and in the case of **anti-synchronization** with the settlement to zero of the error functions $e_i = x_i + \tilde{x}_i, i = 1 \div 5$ the variables of one system will become the same by module, but with the opposite signs, of the variables of the other system. **Hybrid synchronization** is a kind of combination of error functions of the type (6) and (7), and it is chosen randomly which functions e_i to be in the form of differences between the respective pairs of variables and which - in the form of sums of the other pairs of variables of both systems.

The synthesis of the control functions $g_i(\mathbf{x},\tilde{\mathbf{x}})$ to the equations of the slave system is based on the principle of proving the stability of the point $\mathbf{e} = 0$ taking into account the exact type of functions of this vector - (6), (7), or a combination for hybrid synchronization. The problem is solved on the basis of the fulfillment of the conditions for stability of a system by the second Lyapunov method, according to which, if a function $V(\mathbf{e})$ called Lyapunov function is found, which meets the following conditions:

$$V(\mathbf{e}) > 0,\ \forall \mathbf{e} \neq 0, \tag{8}$$

$$V(\mathbf{e}) = 0,\ \mathbf{e} = 0, \tag{9}$$

$$\frac{dV(\mathbf{e})}{dt} < 0,\ \forall \mathbf{e} \neq 0, \tag{10}$$

then the point $\mathbf{e} = 0$ will be stable, which is a sufficient condition for achieving the desired type of synchronization.

Usually a quadratic function $V(\mathbf{e})$ is chosen for such tasks, and for the synchronization system (1) - (3) it will have the form:

$$V(\mathbf{e}) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2\right). \tag{11}$$

The function (11) satisfies the conditions (8) and (9). The derivative of (11) is:

$$\frac{dV(\mathbf{e})}{dt} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5. \tag{12}$$

If the first derivatives of the error functions are in the form:

$$\begin{aligned}
\dot{e}_1 &= -k_1 e_1,\\
\dot{e}_2 &= -k_2 e_2,\\
\dot{e}_3 &= -k_3 e_3,\\
\dot{e}_4 &= -k_4 e_4,\\
\dot{e}_5 &= -k_5 e_5,
\end{aligned} \tag{13}$$

then the expression (12) will become:

$$\frac{dV(\mathbf{e})}{dt} = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 - k_5e_5^2, \tag{14}$$

which is a negatively determined function for $\forall k_i > 0$ and thus the last condition (10) of the second Lyapunov stability method will be satisfied.

In this case, the control functions $g_i(\mathbf{x},\tilde{\mathbf{x}})$ to the system (3) will be chosen in such way that the expressions (13) to be obtained after obtaining the equations of the error system. The type of the error functions - (6), (7) or a combination for hybrid synchronization must be taken into account.

Case 1 – identical synchronization

In this case, the error functions are in the form:

$$e_i = x_i - \tilde{x}_i,\ i = 1 \div 5, \tag{15}$$

and when differentiating the left and right parts of this equation, for the derivatives of the error function holds the following expression:

$$\dot{e}_i = \dot{x}_i - \dot{\tilde{x}}_i,\ i = 1 \div 5\,. \tag{16}$$

Obviously, the expressions (16) can be obtained by subtracting the corresponding equations of (3) from those of (1). The resulting system of equations is called the error system of the synchronization scheme:

$$\begin{aligned}
\dot{e}_1 &= \dot{x}_1 - \dot{\tilde{x}}_1 = a_1 x_1 + a_2 x_2 x_3 + a_3 x_4 x_5 - \\
&\quad - a_1 \tilde{x}_1 - a_2 \tilde{x}_2 \tilde{x}_3 - a_3 \tilde{x}_4 \tilde{x}_5 - g_1(\mathbf{x}, \tilde{\mathbf{x}}), \\
\dot{e}_2 &= \dot{x}_2 - \dot{\tilde{x}}_2 = a_4 x_2 + a_5 x_1 x_3 - \\
&\quad - a_4 \tilde{x}_2 - a_5 \tilde{x}_1 \tilde{x}_3 - g_2(\mathbf{x}, \tilde{\mathbf{x}}), \\
\dot{e}_3 &= \dot{x}_3 - \dot{\tilde{x}}_3 = a_6 x_3 + a_7 x_1 x_2 + r - \\
&\quad - a_6 \tilde{x}_3 - a_7 \tilde{x}_1 \tilde{x}_2 - r - g_3(\mathbf{x}, \tilde{\mathbf{x}}), \\
\dot{e}_4 &= \dot{x}_4 - \dot{\tilde{x}}_4 = a_8 x_4 - x_1 x_5 - \\
&\quad - a_8 \tilde{x}_4 + \tilde{x}_1 \tilde{x}_5 - g_4(\mathbf{x}, \tilde{\mathbf{x}}), \\
\dot{e}_5 &= \dot{x}_5 - \dot{\tilde{x}}_5 = -x_5 + a_9 x_1 x_4 + \\
&\quad + \tilde{x}_5 - a_9 \tilde{x}_1 \tilde{x}_4 - g_5(\mathbf{x}, \tilde{\mathbf{x}}).
\end{aligned} \tag{17}$$

In order to bring the error system (17) into the form (13) and to satisfy the condition (10), and taking into account the expressions (15), the control functions $g_i(\mathbf{x}, \tilde{\mathbf{x}})$ are synthesized in the form:

$$\begin{aligned}
g_1(\mathbf{x}, \tilde{\mathbf{x}}) &= a_1 e_1 + a_2 x_2 x_3 + a_3 x_4 x_5 - \\
&\quad - a_2 \tilde{x}_2 \tilde{x}_3 - a_3 \tilde{x}_4 \tilde{x}_5 + k_1 e_1, \\
g_2(\mathbf{x}, \tilde{\mathbf{x}}) &= a_4 e_2 + a_5 x_1 x_3 - \\
&\quad - a_5 \tilde{x}_1 \tilde{x}_3 + k_2 e_2, \\
g_3(\mathbf{x}, \tilde{\mathbf{x}}) &= a_6 e_3 + a_7 x_1 x_2 - \\
&\quad - a_7 \tilde{x}_1 \tilde{x}_2 + k_3 e_3, \\
g_4(\mathbf{x}, \tilde{\mathbf{x}}) &= a_8 e_4 - x_1 x_5 - \\
&\quad + \tilde{x}_1 \tilde{x}_5 + k_4 e_4, \\
g_5(\mathbf{x}, \tilde{\mathbf{x}}) &= -e_5 + a_9 x_1 x_4 - \\
&\quad - a_9 \tilde{x}_1 \tilde{x}_4 + k_5 e_5,
\end{aligned} \tag{18}$$

where k_i must be positive constants for the expression (14) to be negatively defined.

The synchronization system (1) - (3) with the control functions for identical synchronization (18) is simulated with randomly selected initial conditions of the two systems, respectively $\mathbf{x}(0) = [5 \quad 4 \quad 1 \quad 1 \quad 1]^T$ and $\tilde{\mathbf{x}}(0) = [4 \quad 5 \quad 2 \quad 3 \quad 0.5]^T$. In Fig. 4 the functions $e_i = x_i - \tilde{x}_i,\ i = 1 \div 5$ for $k_i = 1,\ i = 1 \div 5$ are shown. It can be seen that after a transient process of 5s the systems (1) and (3) become identically synchronized. Changing the initial conditions does not affect the synchronization. When selecting larger values of the coefficients k_i, the functions (15) tend faster to zero.

A better idea of the nature of identical synchronization can be obtained if the time evolutions of the variables of systems (1) and (3) are observed together. For example, the joint time series of $x_2(t), \tilde{x}_2(t)$ and $x_4(t), \tilde{x}_4(t)$ are shown in Fig. 5. The two systems start from different initial conditions, but after the end of the transient process each variable from the system (3) performs identical movements to the corresponding variable from the system (1), and these movements remain chaotic. It is this property of chaotic synchronization that underlies the implementation of communication systems with chaotic information protection.

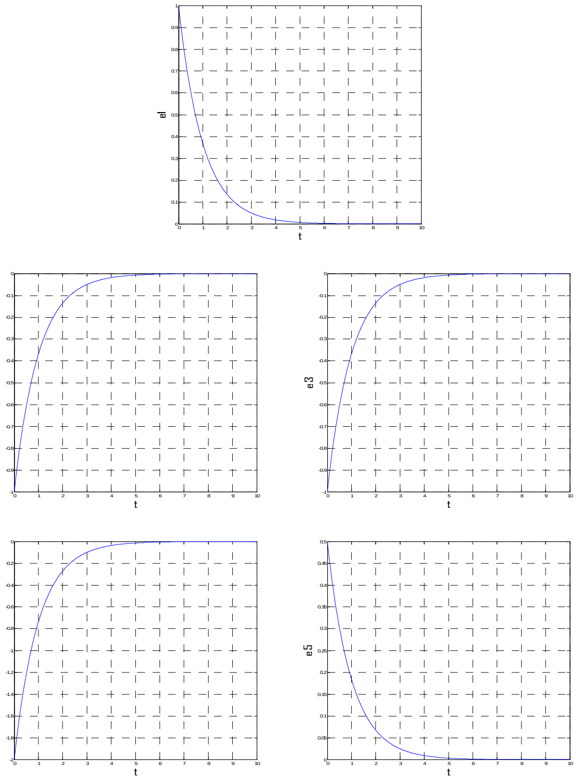

Fig. 4. Error functions (15) in the case of identical synchronization

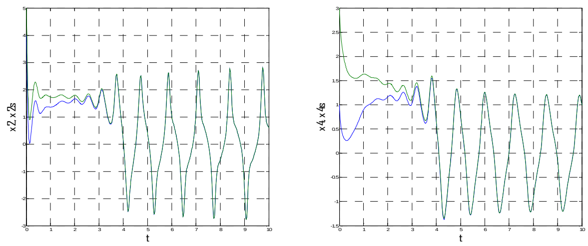

Fig. 5. Joint dynamics of some pairs of state variables of the systems (1) and (3) by identical synchronization

Case 2 – anti-synchronization

In this case, the error functions are in the form:

$$e_i = x_i + \tilde{x}_i,\ i = 1 \div 5\,, \tag{19}$$

and their derivatives are:

$$\dot{e}_i = \dot{x}_i + \dot{\tilde{x}}_i,\ i = 1 \div 5\,. \tag{20}$$

The expressions of the error system (20) will be obtained by adding the equations of the system (1) to the corresponding equations of the system (3):

$$\begin{aligned}
\dot{e}_1 &= \dot{x}_1 + \dot{\tilde{x}}_1 = a_1 x_1 + a_2 x_2 x_3 + a_3 x_4 x_5 + \\
&\quad + a_1 \tilde{x}_1 + a_2 \tilde{x}_2 \tilde{x}_3 + a_3 \tilde{x}_4 \tilde{x}_5 + g_1(\mathbf{x}, \tilde{\mathbf{x}}), \\
\dot{e}_2 &= \dot{x}_2 + \dot{\tilde{x}}_2 = a_4 x_2 + a_5 x_1 x_3 + \\
&\quad + a_4 \tilde{x}_2 + a_5 \tilde{x}_1 \tilde{x}_3 + g_2(\mathbf{x}, \tilde{\mathbf{x}}), \\
\dot{e}_3 &= \dot{x}_3 + \dot{\tilde{x}}_3 = a_6 x_3 + a_7 x_1 x_2 + r + \\
&\quad + a_6 \tilde{x}_3 + a_7 \tilde{x}_1 \tilde{x}_2 + r + g_3(\mathbf{x}, \tilde{\mathbf{x}}), \\
\dot{e}_4 &= \dot{x}_4 + \dot{\tilde{x}}_4 = a_8 x_4 - x_1 x_5 + \\
&\quad + a_8 \tilde{x}_4 - \tilde{x}_1 \tilde{x}_5 + g_4(\mathbf{x}, \tilde{\mathbf{x}}), \\
\dot{e}_5 &= \dot{x}_5 + \dot{\tilde{x}}_5 = -x_5 + a_9 x_1 x_4 - \\
&\quad - \tilde{x}_5 + a_9 \tilde{x}_1 \tilde{x}_4 + g_5(\mathbf{x}, \tilde{\mathbf{x}}).
\end{aligned} \tag{21}$$

In order to bring the system (17) into the form (13) and to satisfy the condition (10), and taking the expressions (19) into account, the control functions $g_i(\mathbf{x},\tilde{\mathbf{x}})$, are synthesized in the form:

$$\begin{aligned}
g_1(\mathbf{x},\tilde{\mathbf{x}}) &= -a_1e_1 - a_2x_2x_3 - a_3x_4x_5 - \\
&\quad - a_2\tilde{x}_2\tilde{x}_3 - a_3\tilde{x}_4\tilde{x}_5 - k_1e_1, \\
g_2(\mathbf{x},\tilde{\mathbf{x}}) &= -a_4e_2 - a_5x_1x_3 - \\
&\quad - a_5\tilde{x}_1\tilde{x}_3 - k_2e_2, \\
g_3(\mathbf{x},\tilde{\mathbf{x}}) &= -a_6e_3 - a_7x_1x_2 - 2r - \\
&\quad - a_7\tilde{x}_1\tilde{x}_2 - k_3e_3, \\
g_4(\mathbf{x},\tilde{\mathbf{x}}) &= -a_8e_4 + x_1x_5 + \\
&\quad + \tilde{x}_1\tilde{x}_5 - k_4e_4, \\
g_5(\mathbf{x},\tilde{\mathbf{x}}) &= e_5 - a_9x_1x_4 - \\
&\quad - a_9\tilde{x}_1\tilde{x}_4 - k_5e_5,
\end{aligned} \tag{22}$$

where k_i must be positive constants again.

The synchronization system (1) - (3) with control functions for anti-synchronization (22) is simulated with the same initial conditions and values of k_i as in Case 1. Fig. 6 shows two of the error functions (19) - $e_1(t)$ and $e_5(t)$, which confirm the existence of anti-synchronization, since this time the sums of the respective variables of (1) and (3) tend to zero.

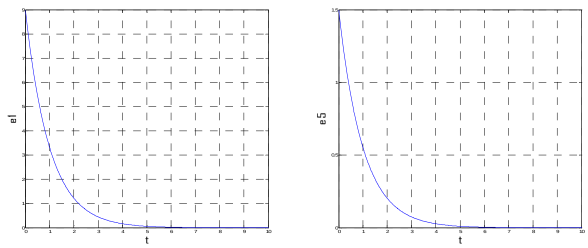

Fig. 6. Some of the error functions (19) in the case of anti-synchronization

Fig. 7 shows the joint time series of the pairs $x_3(t), \tilde{x}_3(t)$ and $x_5(t), \tilde{x}_5(t)$ which illustrate the essence of the anti-synchronization of chaotic systems - after the end of the transient process the given variable from the system (3) has a motion, which is symmetric with respect to the abscissa axis of the motion of the corresponding variable from the system (1).

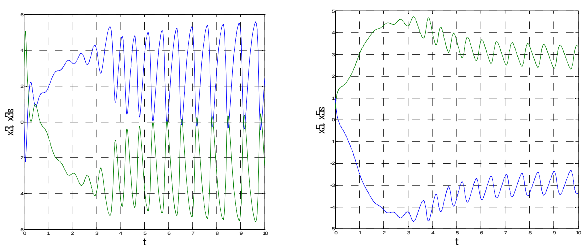

Fig. 7. Joint dynamics of some pairs of variables of the systems (1)and (3) by anti-synchronization

A good idea of the anti-synchronization phenomenon is also obtained from the three-dimensional projections of the chaotic attractors of the master and the slave systems. Fig. 8 shows two of the three-dimensional projections of the five-dimensional chaotic attractor in the subspaces of the state space (x_2, x_3, x_5) and (x_1, x_3, x_4).

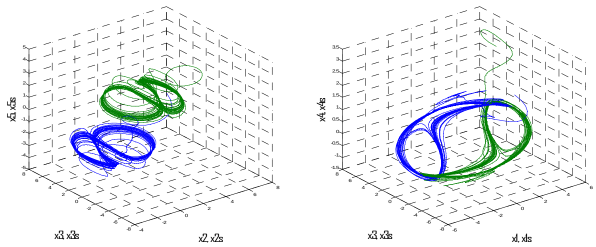

Fig. 8. Projections of the attractors of the systems (1)and (3) in different subspaces of the state space in the case of anti-synchronization

Case 3 – hybrid synchronization

In the identical synchronization case, the error functions are: $e_i = x_i - \tilde{x}_i,\ i = 1 \div 5$, and in the anti-synchronization case they are: $e_i = x_i + \tilde{x}_i,\ i = 1 \div 5$. In hybrid synchronization case, some part of the error functions e_i are chosen to be the differences, and the rest are chosen to be the sums of the respective pairs of variables of the two systems (1) and (3), i.e. many combinations are possible. Let the following combination of error functions for hybrid synchronization be selected:

$$\begin{aligned}
e_1 &= x_1 - \tilde{x}_1, \\
e_2 &= x_2 + \tilde{x}_2, \\
e_3 &= x_3 - \tilde{x}_3,, \\
e_4 &= x_4 + \tilde{x}_4 \\
e_5 &= x_5 - \tilde{x}_5.
\end{aligned} \tag{23}$$

The expressions for the derivatives of the functions (23) are similar. They are obtained after subtraction, respectively summation of the respective equations of the master system (1) and the slave system (3) according to the chosen type of error functions (23):

$$\begin{aligned}
\dot{e}_1 &= \dot{x}_1 - \dot{\tilde{x}}_1 = a_1x_1 + a_2x_2x_3 + a_3x_4x_5 - \\
&\quad - a_1\tilde{x}_1 - a_2\tilde{x}_2\tilde{x}_3 - a_3\tilde{x}_4\tilde{x}_5 - g_1(\mathbf{x},\tilde{\mathbf{x}}), \\
\dot{e}_2 &= \dot{x}_2 + \dot{\tilde{x}}_2 = a_4x_2 + a_5x_1x_3 + \\
&\quad + a_4\tilde{x}_2 + a_5\tilde{x}_1\tilde{x}_3 + g_2(\mathbf{x},\tilde{\mathbf{x}}), \\
\dot{e}_3 &= \dot{x}_3 - \dot{\tilde{x}}_3 = a_6x_3 + a_7x_1x_2 + r - \\
&\quad - a_6\tilde{x}_3 - a_7\tilde{x}_1\tilde{x}_2 - r - g_3(\mathbf{x},\tilde{\mathbf{x}}), \\
\dot{e}_4 &= \dot{x}_4 + \dot{\tilde{x}}_4 = a_8x_4 - x_1x_5 + \\
&\quad + a_8\tilde{x}_4 - \tilde{x}_1\tilde{x}_5 + g_4(\mathbf{x},\tilde{\mathbf{x}}), \\
\dot{e}_5 &= \dot{x}_5 - \dot{\tilde{x}}_5 = -x_5 + a_9x_1x_4 + \\
&\quad + \tilde{x}_5 - a_9\tilde{x}_1\tilde{x}_4 - g_5(\mathbf{x},\tilde{\mathbf{x}}).
\end{aligned} \tag{24}$$

Then, in order to bring the error system for the selected case (24) in the form of (13) and to satisfy the condition (10), and taking into consideration the chosen type of error functions (23), the control functions $g_i(\mathbf{x},\tilde{\mathbf{x}})$ must be synthesized in the form:

$$\begin{aligned}
g_1(\mathbf{x},\tilde{\mathbf{x}}) &= a_1e_1 + a_2x_2x_3 + a_3x_4x_5 - \\
&\quad - a_2\tilde{x}_2\tilde{x}_3 - a_3\tilde{x}_4\tilde{x}_5 + k_1e_1, \\
g_2(\mathbf{x},\tilde{\mathbf{x}}) &= -a_4e_2 - a_5x_1x_3 - \\
&\quad - a_5\tilde{x}_1\tilde{x}_3 - k_2e_2, \\
g_3(\mathbf{x},\tilde{\mathbf{x}}) &= a_6e_3 + a_7x_1x_2 - \\
&\quad - a_7\tilde{x}_1\tilde{x}_2 + k_3e_3, \\
g_4(\mathbf{x},\tilde{\mathbf{x}}) &= -a_8e_4 + x_1x_5 + \\
&\quad + \tilde{x}_1\tilde{x}_5 - k_4e_4, \\
g_5(\mathbf{x},\tilde{\mathbf{x}}) &= -e_5 + a_9x_1x_4 - a_9\tilde{x}_1\tilde{x}_4 + k_5e_5.
\end{aligned} \tag{25}$$

The hybrid synchronization obtained by applying the control functions (25) to the slave system (3) is illustrated in Fig. 9 with the projections of the five-dimensional state space in the subspaces (x_1,x_2,x_3) and (x_1,x_3,x_4). The first figure clearly shows the identical movement of the systems along the axes x_1 and x_3 and the symmetrical movement with respect to the axis x_2, which is in accordance with the selected error functions (23). Similarly, in the second figure the motions of the systems (1) and (3) along the axes x_1 and x_3 are identical, and with respect to the axis x_4 the motions are symmetrical, also in accordance with (23).

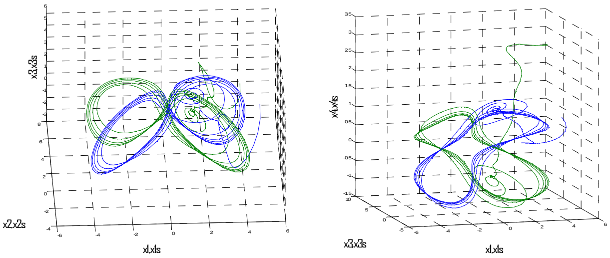

Fig. 9. Projections of the attractors of the systems (1) and (3) in different subspaces of the state space in the case of hybrid synchronization

Fig. 10 shows the attractors of systems (1) and (3) in the projection (x_2,x_3,x_4) of the five-dimensional state space from different points of view. In accordance with the selected error functions (23), the movement of the two systems along the axis x_1 is identical, and with respect to the axes x_2 and x_4 - symmetrical.

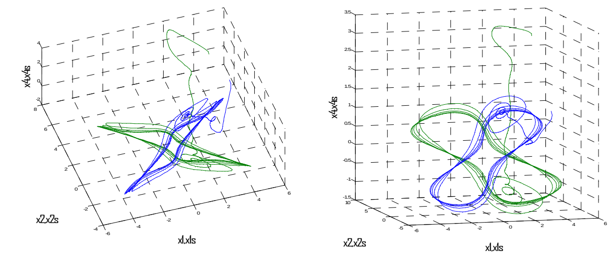

Fig. 10. Projections of the attractors of the systems (1) and (3) in the subspace (x_2,x_3,x_4) in the case of hybrid synchronization

All results presented in the article are obtained for the same values of the coefficients of the control functions k_i. In additional experiments, it was found that these coefficients can be used to adjust the duration of the transient synchronization process.

CONCLUSION

The article illustrates the possibility of obtaining complex types of chaotic synchronization between two chaotic systems of relatively high order. Unlike the chaotic third- and somewhat fourth-order models, very few chaotic systems of the fifth or higher order are known. At the same time, such systems have interesting and, above all, complex dynamics and their analysis is of practical interest in order to implement chaotic synchronization schemes that can be used in secure communications.

The possibility for realization of synchronization schemes for the fifth-order model with different types of chaotic synchronization - identical, anti- and hybrid synchronization is shown. The classical method of Lyapunov stability analysis is used as a basis for the design of the control functions. The effectiveness of the synthesized functions is tested at the simulation level. The obtained results clearly illustrate the obtaining of the desired type of synchronization between the systems. In experiments with different initial conditions, it was found that the initial conditions did not affect the duration of the transient process or the type of synchronization.

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