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MATHEMATICAL MODEL FOR A PNEUMATIC FORCE ACTUATOR SYSTEM

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Abstract

We consider pneumatic cylinder tubes that connect valves with actuators. The aim of this work is to obiaun a new mathematical model for online control applications. For that purpose, a second-order hyperbolic equation corresponding to the model problem is derived. The Laplace transform is used and by this approach a sufficiently simple and applicable model is derived. The effect of different mathematical models for the considered problem is also discussed.

Finally, some validation experiments by real engineering parameters are presented.

Keywords: pneumatic force, mass flow, hyperbolic equation

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1. INTRODUCTION AND NOTATIONS

The pneumatic cylinders can propose a better alternative to electrical or hydraulic actuators for certain types of applications. It could be cited, for example: low cost; clean environmentals; easier to work with and so on.

One of the main problems is to investigate the effect of time delay and attenuation due to the connecting tubes. So (see [1]), the pressure drop along the tube induces a decrease in the air flow through the valve.

On the other hand, the flow at the outlet will be delayed with respect to the one at the inlet by the time increment necessary for the acoustic wave. Thus, the problem of the pressure losses and time delay in long pneumatic lines has to be analyzed. The investigations are based on the assumption of fully developed laminar flow through the tube (see, e.g. [1,2]). In order to find an expression for the mass flow, we also extend our analysis to include wholly turbulent flow.

For a cylindrical tube with length L_t we adopt the following (standard) notations:

- P is the pressure along the tube;
- v is the velocity;
- ρ is the air dencity;
- c denotes the sound speed.

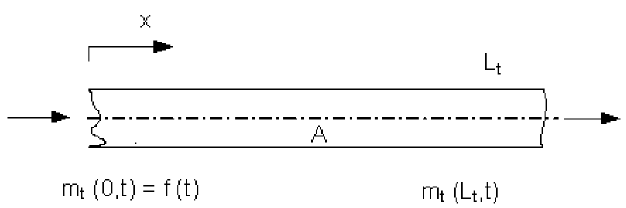

Fig. 1: Pneumatic cylinder notations

Let also x and t be the tube axis and the time variable, respectively (see Fig. 1).

We denote by R the tube resistance and A is the tube cross-sectional area.

In general, considering the control volume V, by standard arguments, the mass m is represented by $m = \rho V$ and then the mass flow rate can be expressed as:

$$\dot{m} = m_t = \frac{d}{dt}(\rho V).$$

For our case, we introduce the mass flow through the tube (one-dimensional) as:

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$$m_t = \rho A v. \quad (1)$$

2. MAIN RESULTS

It is well-known (see, e.g. [1,3]), that the basic equations governing the flow in a circular pneumatic line are written as:

$$\frac{\partial P}{\partial x} = -R v - \rho \frac{\partial v}{\partial t}$$
$$\frac{\partial v}{\partial x} = -\frac{1}{\rho c^2} \frac{\partial P}{\partial t}. \quad (2)$$

Using (1), these equations could be easily transformed:

$$\frac{\partial P}{\partial x} = -\frac{1}{A}\frac{\partial m_t}{\partial t} - \frac{P}{\rho A} m_t$$
$$\frac{\partial m_t}{\partial x} = -\frac{A}{c^2}\frac{\partial P}{\partial t}. \quad (3)$$

Differentiating the first equation of (3) with respect to t and the second one with respect of x, the pressure P is eliminated. So that, the main system (2) is transformed as one equation for the mass flow through the tube:

$$\frac{\partial^2 m_t}{\partial t^2} - c^2 \frac{\partial^2 m_t}{\partial x^2} + \frac{R}{\rho}\frac{\partial m_t}{\partial t} = 0. \quad (4)$$

This differential equation is of hyperbolic type.

We put

$$m_t(x,t) = e^{kt} u(x,t),$$

where $u(x,t)$ is unknown function and k is a parameter.

We choose k in such a way that the resulting equation with respect to $u(x,t)$ contains no first derivative term.

We calculate:

$$\frac{\partial^2 m_t}{\partial x^2} = e^{kt}\frac{\partial^2 u}{\partial x^2};\ \frac{\partial m_t}{\partial t} = e^{kt}\left(ku + \frac{\partial u}{\partial t}\right)$$

and

$$\frac{\partial^2 m_t}{\partial t^2} = e^{kt}\left(k^2 u + 2k\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2}\right).$$

Then, from (4) it follows:

$$e^{kt}\left(k^2 u + 2k\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2}\right) - c^2 e^{kt}\frac{\partial^2 u}{\partial x^2} + e^{kt}\frac{R}{\rho}\left(ku + \frac{\partial u}{\partial t}\right) = 0.$$

Dividing by the exponential multiplier, we obtain:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + \left(k^2 + \frac{R}{\rho}k\right)u + \left(2k + \frac{R}{\rho}\right)\frac{\partial u}{\partial t} = 0.$$

So, we determine the parameter k:

$$2k + \frac{R}{\rho} = 0,$$

or $k = -\frac{R}{2\rho}$ and (4) is transformed in the form:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} - \left(\frac{R}{2\rho}\right)^2 u = 0. \quad (5)$$

This is a wave equation (one-dimensional). The dissipative term $\left(\frac{R}{2\rho}\right)^2 u$ causes that the solution waves do not propagate with the same velocity. Obviously, we have an asymptotic condition: $u(x,t) \to 0$, when x goes to infinity.

The boundary condition is:

$$u(0,t) = h(t),$$

where $h(t)$ is a given function.

Knowing the input mass flow (see Figure 1):

$$f(t) = e^{-\frac{R}{2\rho}t} h(t).$$

Our model problem admits also two homogeneous initial conditions:

$$u(x,0) = 0,$$
$$\frac{\partial u}{\partial t}(x,0) = 0. \quad (7)$$

Using the Laplace transform, we get:

$$u(x,t) \to \bar{u}(x,s) = L(u(x,t)),$$

where L is an integral operator and s is the Laplace parameter (see, e.g. [4]).

Then $\frac{\partial^2 u}{\partial x^2} \leftrightarrow \frac{d^2 \bar{u}}{dx^2}$.

Using the initial conditions (7), we obtain $\frac{\partial^2 u}{\partial t^2} \leftrightarrow s^2 \bar{u}(x,s)$ and consequently (5) is transformed in the following linear ordinary differential equation:

$$c^2\frac{d^2\bar{u}}{dx^2} - s^2\bar{u} + \frac{R^2}{4\rho^2}\bar{u} = 0,$$

or,

$$\frac{d^2\overline{u}}{dx^2}-\frac{s^2}{c^2}\left(1-\frac{R^2}{4\rho^2 s^2}\right)\overline{u}=0. \qquad (8)$$

First, let us denote $\frac{R}{2\rho}=\varepsilon>0$. Then the general solution of (8) is:

$$\overline{u}(x,s)=C_1\, e^{-\frac{s}{c}x\sqrt{1-\frac{\varepsilon^2}{s^2}}}+C_2\, e^{\frac{s}{c}x\sqrt{1-\frac{\varepsilon^2}{s^2}}}.$$

From the asymptotic property, it follows that ($C_1>0$):

$$\overline{u}(x,s)=C_1\, e^{-\frac{s}{c}x\sqrt{1-\frac{\varepsilon^2}{s^2}}}.$$

Now, we use the expansion ($\frac{\varepsilon^2}{s^2}$ approximates the zero):

$$\sqrt{1-\frac{\varepsilon^2}{s^2}}\approx 1-\frac{\varepsilon^2}{2s^2}.$$

Then

$$\overline{u}(x,s)=C_1\, e^{-\frac{s}{c}x\left(1-\frac{\varepsilon^2}{2s^2}\right)}.$$

Let us emphasize that $C_1=C_1(s)$

If $x = 0$, from (6) we have:

$$C_1(s)=\overline{h}(s)=\int_0^\infty e^{-st}\,h(t)\,dt,$$

hence

$$\overline{u}(x,s)=\overline{h}(s)\,e^{-\frac{x}{c}s}\,e^{\frac{\varepsilon^2 x}{2cs}}.$$

Using again "first approximation" of the exponential function, for the inversion $\overline{u}(x,s)$ we obtain:

$$\overline{u}(x,s)=e^{-\frac{x}{c}s}\,\overline{h}(s)\left(1+\frac{\varepsilon^2 x}{2cs}\right).$$

From the delay rule [4], the solution for this boundary-initial-value problem is:

$$u(x,t)=\begin{cases}0, & t\le\frac{x}{c},\\ h\left(t-\frac{x}{c}\right)\left[\delta\left(t-\frac{x}{c}\right)+\frac{\varepsilon^2 x}{2c}H\left(t-\frac{x}{c}\right)\right], & t>\frac{x}{c},\end{cases}$$

when δ is the Dirak function and H is the Heaviside unit function. Here $t-\frac{x}{c}$ is a retardant argument.

The input wave will rich the end of the tube in a time period $\tau=\frac{L_t}{c}$.

Then

$$m_t(L_t,t)=e^{-\frac{R}{2\rho}t}\,u(L_t,t).$$

On the other hand,

$$h\left(t-\frac{L_t}{c}\right)=e^{\frac{R}{2\rho}\left(t-\frac{L_t}{c}\right)}\,f\left(t-\frac{L_t}{c}\right).$$

Finally, if $x=L_t$ and $t>\frac{L_t}{c}$, the mass flow at the outlet of the tube is:

$$m_t=\frac{R^2\,L_t}{8c\,\rho^2}\,e^{-\frac{R\,L_t}{2\rho c}}\,f\left(t-\frac{L_t}{c}\right). \qquad (9)$$

It could be replaced the air density ρ by the pressure and temperature from the equation of state.
Namely,

$$P=\rho\,R_0\,T,$$

where R_0 is the ideal gas constant and T is the temperature.

Equation (9) describes in a simple form the mass flow at the tube outlet.

3. EXPERIMENTAL RESULTS

In order to illustrate the presented mathematical model, we performed a simple experiment using a plastic cylindrical tube with *8* mm internal diameter and *5* m length. The input flow (input function *f(t)*) was constant.

Fig. 2 shows measured values of input and outlet flow in dimensionless form. One can observe that the proposed method can effectively approximate the outlet flow as well as one can predict the time delay.

Fig. 3 represents the corresponding measured initial and outlet pressures for the same tube.

The experimental parameters we have used are: $R = 1.37$; $\rho = 1.21$ kg/m^3; $C = 343$ m/s.

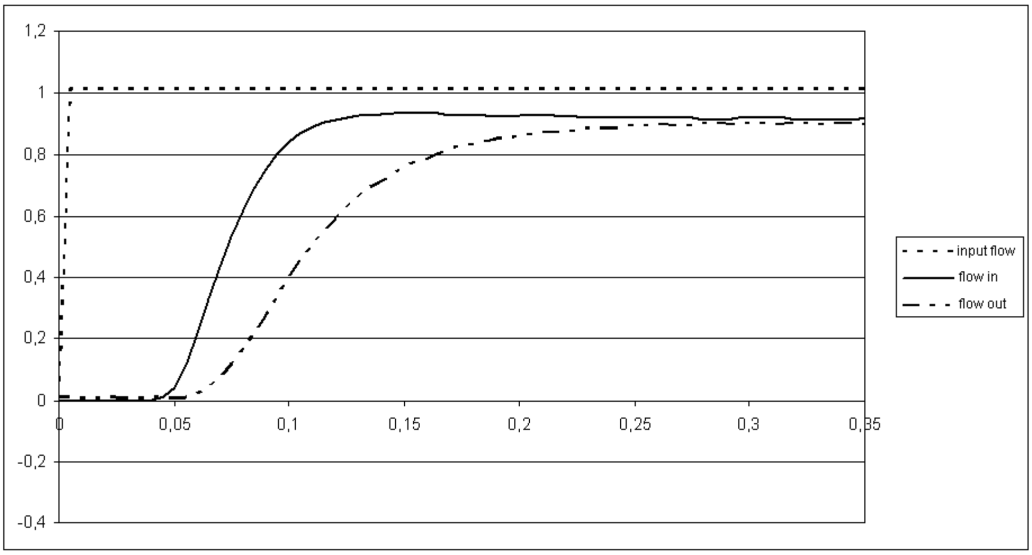

Fig. 2: Tube outlet flow for a constant flow input

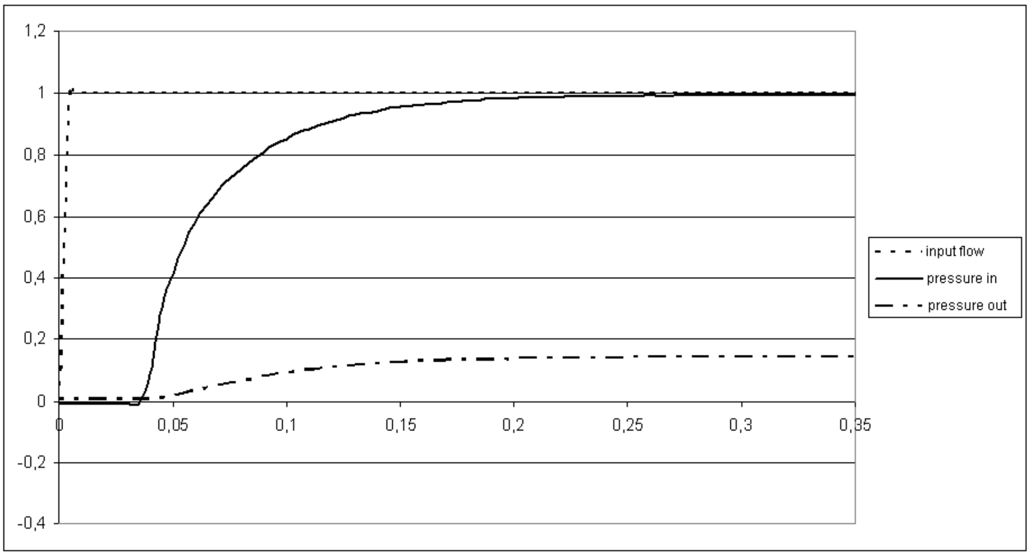

Fig. 3: Flow pressure

4. CONCLUSIONS

In this paper we developed a detailed mathematical model for a pneumatic cylinder tube that connected the valve with the actuator. The proposed model is sufficiently simple, such that it can be used online in control applications.

The formula of the mass flow shows that the flow profile at the outlet will be delayed with respect to the one at the inlet by the time increment for the wave to travel the entire length of the tube. So, we are able to determine the effective tube length L_t connected with the time delay.

Some model validation experiments are presented.

It is also worth while noting that pneumatic actuator systems could be investigated by the method of the Fourier analysis (see, e.g. [5]). Here, we propose a new approach to the problem.

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